JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 30, 398–400 (1970)

Remarks on the Equations of Langmuir and Blasius

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Submitted by R. Bellman

Received June 16, 1969

Inselberg (1) has shown recently that Langmuir's equation

$$3y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 + 4y\frac{dy}{dt} + y^2 = 1 \tag{1}$$

may be reduced to a first order abelian equation of the second type. Estimates for asymptotic phase-plane solutions may then be obtained in a closed form. In this note, we point out a curious connection between the equations of Blasius and Langmuir. More precisely, it is shown that subject to certain conditions, either equation may be transformed into the other by means of elementary substitutions.

The equation of Blasius (2) is

$$f''' + ff'' = 0, \qquad ' \equiv \frac{d}{dx} \tag{2}$$

and occurs in the investigation of the boundary-layer flow of a viscous fluid past a semiinfinite flat plate which is held at zero angle of attack to the oncoming stream. We note that (2) may be rephrased as the nonlinear integral equation

$$f'' = e^{-\int^x f(s)\,ds} \tag{3}$$

in which it is assumed $f'' \neq 0$.

Consider the effect on (3) of the chain of substitutions defined as follows:

(A) Let $f = z'$, then (3) reduces to

$$z''' = e^{-z}. \tag{4}$$

(B) Let $z' = p(z)$, then (4) reduces to

$$p^2\frac{d^2p}{dz^2} + p\left(\frac{dp}{dz}\right)^2 = e^{-z}. \tag{5}$$

(C) Let $p(z) = q^{2/3}$, then (5) reduces to

$$3q\frac{d^2q}{dz^2} + \left(\frac{dq}{dz}\right)^2 = \frac{9}{2}e^{-z}. \tag{6}$$

(D) Finally, let $q = \pm (3/\sqrt{2})\,Q$ and replace z by t, where $t = -z$, then (6) reduces to

$$3Q\frac{d^2Q}{dt^2} + \left(\frac{dQ}{dt}\right)^2 = e^t. \tag{7}$$

Equation (7) is a form of Langmuir's equation noted by Inselberg and given in Davis (3). The classical form (1) of the equation is obtained by the substitution

$$y = e^{-t/2}Q.$$

Consequently, if $f(x)$ is a known solution of the Blasius equation for which $f''(x) \neq 0$, then a solution of Langmuir's equation is $y(t)$, where

$$y(t) = \pm\frac{\sqrt{2}}{3}e^{-t/2}\{f(x)\}^{3/2}$$

with x defined implicitly in terms of t by means of the relation

$$t = -\int^x f(s)\,ds.$$

A converse result also holds, e.g., the singular solutions of (1),

$$y(t) = \pm 1,$$

correspond to the well-known solution of Blasius' equation

$$f(x) = \frac{3}{\lambda + x}, \qquad \lambda \text{ arbitrary.}$$

We conclude that it may be worthwhile to convert Langmuir's equation to the higher order Blasius equation and make use of the techniques developed for this equation. In particular, one might consider the powerful method of iteration of Weyl (4) and the comprehensive phase-plane analysis of Coppel (5).

References

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